Math 227A: Problem Set 3 and Suggested Exercises for Week 6

The following exercises are Problem Set 3, and are due on May 13:

1. Milnor and Stasheff 2A, 2B, 3A, 3E.

The following are suggested exercises for Week 6:

1. Show that the orthogonal complement of a subbundle is, up to isomorphism, independent of the choice of inner product.

2. Consider the diagonal embedding $\Delta : M \to M \times M$. Construct the tangent bundle and normal bundle of $\Delta \subset M \times M$, and show they are isomorphic.

3. *Cohomology of fibre bundles* Here is an outline for proving the Leray-Hirsch theorem: Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibre bundle. Suppose that for some coefficient ring R, $H^n(F;R)$ is a finitely generated free R-module for all n and there are classes $c_j \in H^*(E;R)$ whose restrictions form a basis for $H^*(F;R)$ for each fibre F. Then $H^*(E;R) \simeq H^*(F;R) \otimes H^*(B;R)$.

   - Start with a CW complex B. If B is zero-dimensional this is trivial, so by induction, assume we already know the result for $(n-1)$-dimensional complexes. Let B be n-dimensional, and B' be the space obtained by deleting a single point x_α from the interior of each n-cell. Let $E' = p^{-1}(B')$. Show there is a commutative diagram of exact sequences

$$\begin{array}{ccccccc} \longrightarrow H^*(B,B';R) \otimes H^*(F;R) & \longrightarrow & H^*(B;R) \otimes H^*(F;R) & \longrightarrow & H^*(B';R) \otimes H^*(F;R) \\ \downarrow & & \downarrow & & \downarrow \\ \longrightarrow H^*(E,E';R) & \longrightarrow & H^*(E;R) & \longrightarrow & H^*(E';R) \longrightarrow \end{array}$$

   - Show that the inclusion $p^{-1}(B^{n-1}) \to E'$ is a weak homotopy equivalence, which establishes that the rightmost map above is an isomorphism.
   - Use excision and the Kunneth formula to show the leftmost map is an isomorphism (Hint: near each x_α, the fibre bundle is just a product).
   - Now let B be a potentially infinite CW complex. Use n-connectivity of (B, B^n) to extend the result to a fibration with base B.
   - Extend to arbitrary bases B by using CW approximation and considering the pullback bundle.

4. Hatcher 4.D.1, 4.D.2 (applications of the above, including an example where the identification need not be a ring isomorphism).